ON FOUNDATION OF FUZZY DECISION MAKING

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Abstract. Bellman and Zadeh elaborated one of the methods of decision making in a fuzzy environment. However, its application correctness for some applied problems remains unclarified. The point of the given paper is to dwell on some problems designed to work out an appropriate correctness test scheme. Zimmerman et al. have shown the application limits of the Bellman-Zadeh method with the use of one of the minimum or product functions. The Bellman-Giertz theorem is based on the correctness analysis. It is suggested to analyze the validity of the Bellman-Giertz axioms for concrete decision making problems instead of the experimental test of the minimum function correctness. When problems are found inconsistent with these axioms due to dependability of the arguments, a scheme is proposed to ensure correctness by expanding the argument set of the membership function. In order to obtain such argument set for minimum number of steps in the sense of the Shannon's function a certain statement has been proved and an algorithm and program elaborated on the basis of the given statement. The algorithm utilized the Hancel's lemma for monotone Boolean functions.

Keywords. Decision theory; large-scale systems; mathematical programming; Boolean function; monotone Boolean function; fuzzy sets; dialogue table.

INTRODUCTION

More realistic problem formulation is required for many decision making problems that appear in mathematical programming. Uncertainty of aims and criterions must be obviously considered. R.Bellman and L.Zadeh (1970) have proposed for this purpose a scheme (BZS). It is used in solution of great number of decision making problems but there is a wide disagreement about correctness of its usage to concrete problems. It is not clear yet what problems are related to the sphere of its application. This question may be solved axiomatically (Bellman, Giertz, 1973; Orlov, 1978; Kovalerchuk, 1981) or experimentally (Thole, Zimmermann, Zyspo, 1979).

A membership function of intersection of fuzzy objective functions and fuzzy constraints for real data may differ from minimum or product operators considered by R.Bellman and L.Zadeh.

At present time there is no testing method for correctness of BZS to concrete problems.

The aim of this work is to solve some questions during elaboration of such method. The logic of our approach consists in testing of special systems of axioms for concrete problems. Bellman-Giertz axioms give an example of such axiomatics for operation of minimum (Bellman-Giertz theorem, 1973).

But in comparison with a number of other systems of axioms they have rather definite interpretation. Just this positive property makes it clear that they hardly refer to real problems. In particular, the condition of independence of arguments does ussually not hold.

In this work a correct application of Bellman-Giertz theorem in this situation is proposed, and also some problems of optimal algorithmical and program realization of the method using monotone Boolean functions are solved.

CORRECTNESS PROBLEM

Let us define more precisely the concept of correctness of Bellman-Zadeh's scheme (BZS) for concrete problems.

We suppose, that Θ_M-requirement of BZS (axiomatics of BZS) to the problem Z is given precisely. We also suppose that for the problem Z Θ_Z-its properties (axioms of problem Z) are given precisely.

Then the concept of correct application of BZS to the problem Z may be formalized in the following way.

Definition 1 Application of BZS to the problem Z is called correct, if $\Theta_Z \vdash \Theta_M$, i.e. we have Θ_M from Θ_Z.

For concrete axiomatics a more accurate definition is required for the concept of " $\vdash$ ". In particular this definition may be formulated in classical manner. If an expert knows axiomatics exactly, it is possible to control $\Theta_Z \vdash \Theta_M$

Otherwise, we may put local questions which we may use for control of axioms' correctness Θ_M for concrete problems. For example, it will be necessary to take into consideration conditional probabilities with respect to different particular hypotheses X and Y - P (A/X), P (A/Y), if A exists so that P (A)≠ P(A/X) or P(A)≠ P(A/Y), i.e. A depends on X or Y. Below we analyse Bellman-Gierz's axioms (1973) (BGA). There are other systems of axioms, but BGA is gives more useable minimum operator (Bellman-Giertz theorem). It is very difficult to formulate axioms for uncertain problem Z. One may note that this requirement entails the paradox - uncertain problem and concrete formulation of its properties.

This paradox may be overcome at the first stage of development of approach by the inference to the checking of performance of axioms. Expert answers the question - is an axiom correct in his problem? Hence, it follows that the problem becomes psychological - is the sense of axiom clear to expert and may he correlate axiom with his knowledge in his field. Axioms must be formulated in terms of concrete field, so that expert may answer the question. It requires refusal of formulation of axioms in a usual number forms.

BGA AXIOMS FOR BZS

Bellman, Giertz (1973) gave axioms (BGA) in propositional calculus form (BGA 1) and in number form (BGA 2).

For the real problem solved in dialogue with expert only axioms in propositional calculus terms should be used.

Let us have: set of propositions $V=\{S\}$; set of compound propositions $\Phi(V)=\{\varphi\}$ obtained only from the elements of set $V=\{S\}$ with the help of " $\wedge$ " and " $\vee$ "; fuzzy set (set of fuzzy propositions) $W=\{\varphi, \mu(\varphi)\}$, where $\mu: \Phi(V)\to[0,1]$.

Membership function μ has interpretation as the degree of truth and as the implementation of state described by the proposition. Let us consider axioms - BGA 1 and two properties A) and B) (Bellman, Gierz, 1973). which are the base of axioms. A) and B) are often not considered, but they play an important role for Bellman-Giertz theorem (1973) and for using its result in real, actual problems. They are:

A) $V=\{S: \neg\exists\varphi\in(\Phi(V)\setminus V) \wedge \varphi = S\}$,

i.e. there does not exist $S\in V$ so that S is construction from other propositions (elements of the set V).

B) $\mu(\varphi)=\lambda(\mu(S_1), \mu(S_2), \dots, \mu(S_n))$, where $\varphi=\varphi(S_1, S_2, \dots S_n)$ and λ - is some function into $[0,1]$

It means that membership function value of each compound proposition depends only on the value of membership function of elementary proposition.

Axioms: BGA 1:

1) $\mu(S_1\wedge S_2)=\mu(S_2\wedge S_1)$; $\mu(S_1\vee S_2)=\mu(S_2\vee S_1)$ - commutability;

2) $\mu((S_1\wedge S_2)\wedge S_3)=\mu(S_1\wedge(S_2\wedge S_3))$ $\mu((S_1\vee S_2)\vee S_3)=\mu(S_1\vee(S_2\vee S_3))$ - associativity;

3) $\mu(S_1\wedge(S_2\vee S_3))=\mu((S_1\wedge S_2)\vee(S_1\wedge S_3))$; $\mu((S_1\vee S_2)\wedge(S_1\vee S_3))=\mu(S_1\vee(S_2\wedge S_3))$ - distributivity;

4) $\mu(S_1)\geqslant\mu(S_2) \Rightarrow \mu(S_1\wedge S_3)\geqslant\mu(S_2\wedge S_3)$; $\mu(S_1\vee S_3)\geqslant\mu(S_2\vee S_3)$, where μ - decreasing and continuons;

5) $\mu(S_1)=\mu(S_2) > \mu(S_3)=\mu(S_4) \Rightarrow \mu(S_1\wedge S_2) >$

$\mu(S_3 \wedge S_4)$ -strictly increasing

6) $\mu(S_1 \wedge S_2) \leq \min(\mu(S_1), \mu(S_2))$; $\mu(S_1 \vee S_2) \geq \max(\mu(S_1), \mu(S_2))$ - membership function of conjunction is no more than the membership function of each of components and membership function of disjunction is not less than the membership function of each of components

7) $\mu(S_1) = \mu(S_2) = 1 \Rightarrow \mu(S_1 \wedge S_2) = 1$; $\mu(S_1) = \mu(S_2) = 0 \Rightarrow \mu(S_1 \vee S_2) = 0$

We may consider this axioms as Θ_M. Also this description of Θ_M is more useful than BGA 2 for the dialogue with an expert in non-formalised fields but it is not sufficient to satisfy the whole dialogue.

It is required to define obviously the sets of propositions V, $\mathcal{P}(V)$. A set V may be defined by algebraic system theory. We fix set of one-place predicates Ω, set of subjects X and consider a pair $\langle X, \Omega \rangle$ - a relational system, a model (Malcev, 1970). Then V is a diagram of model in this terms.

It is possible to consider n - 1 based, algebraic system $\langle X, K_1, K_2 \dots K_n, \Omega \rangle$, where Ω is a set of different place predicates.

This concept is natural for fuzzy mathematical programming. It will be shown for one-place predicates. Let us be given $C(X)$-a predicate corresponding to a fuzzy objective function, $G(X)$ - a predicate corresponding to a fuzzy constraint and X - an alternative .

FUZZY MATHEMATICAL PROGRAMMING PROBLEMS

Bellman-Zadeh's extremum problem (1970) was in these terms defined by

$$\max_{x \in X} \mu(C(x) \wedge G(x)) = \max_{x \in X} \min(\mu(C(x)), \mu(G(x))) = \mu_D(x^*)$$

where X^* - maximizing decision. Analogically, the extremum problem for several $C_i(X)$, $G_i(X)$ is defined by

$$\max \mu(C_1^1(X) \wedge C_2^1(X) \wedge \dots \wedge G_2(X) \wedge \dots G_n(X))$$

We may consider two-place, three-place and etc. predicates in these formulae and in Bellman-Giertz axioms as still more realistic approach to applied problems for more detailed description of environment.

Let $T(X,t) = C(X)$, $B(X,b) = G(X)$. Parameters t,b may be fixed or varying.

In these cases maximizing decision is defined by

$$\mu_D(t^*, b^*, x^*) = \max \mu(T(X,t) \wedge b(x,b)) = \max \min(\mu(T(X,t)), \mu B(x,b)))$$

We interpret $T(X,t)$ as "X has the property t", and $B(X,b)$ - "X has the property b or "the alternative X has the characteristic b " Other formulations are possible for $T(X,t)$, $B(X,b)$ in natural languages.

EXAMPLES OF PROBLEMS WITH DEPENDENT PARAMETRES

It is shown below that according to Bellman-Giertz axioms independence of parametres is postulated. But often we encounter dependence.

Example 1

Let: $T(X,t)$ - "The rumen weight of cow X is equal to t"; (in percent)
$B(X,b)$ - "The diet of cow X is b ";
b_1-high forage diet
b_2-high concentrate diet ;
$t_1 = 18\ \%$, $t_2 = 30\ \%$
$\mu(T(X,t))$, $\mu(B(X,b))$ accordingly, membership functions of $T(X,t)$, $B(X,b)$

The problem is to find X so that weight of X is maximal and diet cost of X is minimal.

Let us consider axiom for this example, if

$$\forall x, y, t_\alpha, t_\beta \ \mu(T(x,t_\alpha)) > \mu(T(y,t)) \Rightarrow \mu(T(x,t_\alpha) \wedge B(x,b)) > \mu(T(y,t_\beta) \wedge B(x,b)) \quad (1)$$

According to "British farmer and stockbreeder" vII, 253 06.03.1982 p.19 there is X, such that $\mu(T(x_1,t_1)) > \mu(T(x_1,t_2))$ for $t_\alpha = t_1$, $t_\beta = t_2$, and $\mu(t(X_1,t_2) \wedge B(x,b_2)) < \mu(T(X_1,t_2) \wedge B(X_1, b_1))$ for $b = b_1$, i.e. we have contradiction to the axiom 4. For X_1 t=18% is is preferred over t=30% but B (X, b_1) is less preferable than B(X b_2) i.e. a small rumen does not give full digestibility of forage.

We also have in this case contradiction to the axiom 7, i.e. if $\mu(T(X_1, t_1)) = \mu(B(X_1, b)) = 1$ then $\mu(T(X_1, t_2) \wedge B(X_1, b_1)) \neq 1$

Hence it follows that Bellman-Giertz axioms ignore the dependence of parametres, in particular t and b.

Thus a problem of a regular method for finding independent groups of parameters arises. Dependence of parameters is one of the central problems in foundation of fuzzy optimization.

Thus, Bellman-Giertz's theorem is not correct for dependent parameters and minimum operator is also not correct. The decision for the problem which arises in this situation can be found using the groups of independent parameters in large-scale systems.

FINDING OF GROUPS OF INDEPENDENT PARAMETERS

Let $w(\varphi)$ - be a set of compound propositions, constructed from $S_I, S_2, \ldots S_n$ - components of φ. A length of all propositions φ is less than n, and only "$\wedge$" is used for construction of these compound propositions.

Let us suppose that there exist λ: $\forall \varphi\ \mu(\varphi)=\lambda(\mu(W_1),\ldots,\mu(W_r))$, where $W_i \in W(\varphi)$, i.e. $\mu(\varphi)$ can be described as a function of conjunctions of components with the length less then φ.
.Application of minimum operator is, according to Bellman-Giertz theorem, correct for such λ and φ, if it is correct for I - 7 conditions of the theorem. It is difficult to use function λ with 2^n $(n \geqslant 5)$ arguments. It requires reconstruction of 2^n functions $\mu(W) = \mu(S_I \wedge \wedge S_2 \wedge \ldots \wedge S_r)$.

Hence it follows that there is a question of minimization of the number of arguments of X.

MINIMIZATION OF THE NUMBER OF FUNCTION ARGUMENTS

First a few definitions.

Definition 2 Propositions φ_1, φ_2 are called dependent with respect to a proposition φ_3, μ, if

$\forall S\ \mu(\varphi_1) \geqslant \mu(\varphi_2) \Rightarrow \mu(\varphi_1 \wedge \varphi_3 \wedge S) < \mu(\varphi_2 \wedge \varphi_3 \wedge S)$

or $\mu(\varphi_1) \leqslant \mu(\varphi_2) \Rightarrow \mu(\varphi_1 \wedge \varphi_3 \wedge S) > \mu(\varphi_2 \wedge \varphi_3 \wedge S)$

Example. We illustrate this definition by the example I. Let us suppose that $\varphi_1 = T(X_1, t)$,

$\varphi_2 = T(X_1, t)$, $\varphi_3 = B(X, b)$

According to the definition I

$\mu(\varphi_1) > \mu(\varphi_2) \Rightarrow \forall S\ \mu(\varphi_1 \wedge \varphi_3 \wedge S) < \mu(\varphi_1 \wedge \varphi_2 \wedge S)$

It means that X with a little rumen is preferable than X with a large rumen but a high forage diet and a large rumen is preferable than a high forage diet and a little rumen.

Definition 3. If propositions φ_1, φ_2 are dependent with respect to φ_3 and μ, then propositions are called essential propositions if they are $\varphi_1 \wedge \varphi_3 \wedge S$, $\varphi_2 \wedge \varphi_3 \wedge S$.

Definition 4 A function C(W) such, that C(W)=I if W is an essential proposition and C(W)=0 if W is not an essential proposition is a characteristic function of essentiality.

We now introduce the method of coding propositions W. Let n be the number of S_i and we can consider Bellman n-space $E^n = \{X_i\}$ so that $X_i = X_{iI}, X_{i2}, \ldots, X_{in}$ and $X_{ij}=I$ if and only if S_i is an element of φ.

In this case function C induces Boolean function $\check{C}(x)$: $E^n \to E$, where $E = [0, I]$.

Theorem 1 The function $\check{C}$ is monotone Boolean function.

Proof According to the monotone Boolean function definition it is required to show $\forall\ X_1, X_2 \in E^n$

$X_1 > X_2 \Rightarrow \check{C}(X_1) \geqslant \check{C}(X_2)$.

Let us consider W_I, W_2 corresponding to X_1, X_2. It means that the conjunction W_I includes more components, than W_2^I. Let $C(X)=I$ else verification becomes trivial. It is required to show that $C(X)=I$. If $C(X_2)=I$ then W_2 is an essential conjuction.

We must prove the statement only for $W_I = W_2 \wedge S$ because the proof for $W_I = W_2 \wedge S_1 \wedge S_2 \wedge S_K$ is trivial in this case. Let W_2 be an essential proposition and $W_2 = W_{2I} \wedge W_{22}$.

Hence from definition 3 it follows that $\exists\ W\ \mu(W_{2I}) \geqslant \mu(W) \Rightarrow$

$$\mu(W_{2I} \wedge W_{22}) < \mu(W \wedge W_{22}) \qquad (2)$$

We consider one variant for definition 3, for another variant the proof is analogical.

Let $W_I = W_2 \wedge S = W_{22} \wedge W_{2I} \wedge S$. Then we show that W_I is essential. For this it is sufficient to show

$$\mu(W_{2I}) \geqslant \mu(W) \Rightarrow \mu(W_2 \wedge S) < \mu(W \wedge W_{22} \wedge S) \qquad (3)$$

From definitions 2, 3 and (2)

$\mu(W \wedge W_{22}) > \mu(W_2) \Rightarrow$

$\mu(W_2 \wedge S) < \mu(W \wedge W_{22} \wedge S)$ (4)
comparing (4) and (3), we get the proof of the theorem 2.

Theorem 2 The axiom 4 is an essential proposition .

Proof It is necessary to prove $\mu(W_I) < \mu(W_2) \Rightarrow \forall S$

$\mu(W_I \wedge S) < \mu(W_2 \wedge S)$ (5)

Let $W_I = \varphi_1 \wedge \varphi_3$, $W_2 = \varphi_2 \wedge \varphi_3$ be essential propositions, then by a definitions we have $\mu(\varphi_1) > \mu(\varphi_2) \Rightarrow \forall S \mu(\varphi_1 \wedge \varphi_3 \wedge S) < \mu(\varphi_2 \wedge \varphi_3 \wedge S)$

Suppose that the left hand part of the last implication is true, then $\forall S \mu(W_I \wedge S) < \mu(W_2 \wedge S)$, $\mu(W_I) < \mu(W_2)$ is corresponding to (5).

Hence it follows that the problem of minimization of the number of function's arguments is reduced to the deciphering of monotone Boolean function C (Hancel, 1976).

It is possible to construct, for example, such scheme of reconstruction of monotone Boolean function: an expert answers the question about essentiality proposition given W. Then after the answer on the questions constructed from the essential propositions corresponding to lower units of monotone Boolean function $\check{C}$ (X), a minimal set of λ arguments is be formed.

We may put to expert less questions than 2^n, using monotonicity and lemma (Hancel, 1976), which limit the number of questions by $C_n^a + C_n^b$, where $a = [\frac{n}{2}]$, $b = [\frac{n}{2}] + I$.

We used everything according to Hancel's lemma $\varphi(n) = \min_F \max_f \varphi(F,f) = C_n^a + C_n^b$, where $\{F\}$ – a set of algorithms.

M_n – a set of all monotone Boolean function f, such that $f : E^n \rightarrow E$,

$\varphi(F,f)$ – a number of questions to an expert for deciphering function f by algorithm F , $\varphi(n)$ – Shannon's function.

CONCLUSION

We have suggested algorithm and program (FORTRAN – IV) with the scheme of dialogue described in detail in other paper (Gorbunov, Kovalerchuk, 1982). This program gives the table of dialogue for each n as the kind of scheme of sequence of questions according to the results of previons questions about essentiality of λ arguments.

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