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Explanatory model for the break of logic equivalence by irrational agents in Elkan's Paradox

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Abstract. Fuzzy logic breaks logic equivalence of statements such as $(A\wedge B)\vee(\neg A\wedge B)\vee(A\wedge\neg B)$ and $A\vee B$. It breaks the symmetry of use of such logically equivalent statements. There is a controversy about this property. It is called a paradox (Elkan's paradox) and interpreted as a logical weakness of fuzzy logic. In the opposite view, it is not a paradox but a fundamental postulate of fuzzy logic and one of the sources of its success in applications. There is no explanatory model to resolve this controversy. This paper provides such a model using a vector/matrix logic of rational and irrational agents that covers scalar classical and fuzzy logics. It is shown that the classical logic models rational agents, while fuzzy logic can model irrational agents. Rational agents do not break logic equivalence in contrast with irrational agents. We resolve the paradox by showing that the classical and fuzzy logics have different domains of rational and irrational agents.

Keywords: Fuzzy logic, classical logic, explanatory model, logic equivalence, irrational agent, inconsistent agent, rational agent, paradox

1 Introduction

In classical set theory the union of two sets **A** and **B** is equal to the union of their intersection **A**∩**B** with their proper parts, **A\B** and **B\A,** i.e., **A**∪**B** = (**A**∩**B**) ∪ (**A\B**) ∪ (**B\A**). In classical logic, this property means logic equivalence of truth-values of appropriate statements, $A \vee B = (A\wedge B) \vee (\neg A\wedge B) \vee (A\wedge\neg B)$, but in fuzzy logic this is not true in general,

$$A \vee B \neq (A\wedge B) \vee (\neg A\wedge B) \vee (A \neg B).$$

Fuzzy logic *breaks logic equivalence*. We cannot use $(A\wedge B) \vee (\neg A\wedge B) \vee (A\wedge\neg B)$ instead of $A \vee B$. In other words, fuzzy logic *breaks logic symmetry* of use of these statements. For the first time, the problem of interrelations between equivalence in classic and in fuzzy logic was, discussed in [2] for classical equivalence between disjunction and conjunction normal forms, DNF and CNF.

It is shown in [2] and [5] that not only the DNF and CNF are different in fuzzy logic, but also any classic equivalence is substituted with a class of alternative relations that are not equivalences.

There is a controversy about this break of symmetry property in the literature [6], [7], [13],[14]. It is even called a paradox (Elkan's paradox) and interpreted as a logical weakness of fuzzy logic because it does not follow the classical logic equivalence and related properties. The alternative view is that it is not a paradox, but a fundamental postulate of fuzzy logic and one of the sources of its success in applications. None of the sides provided an *explanatory model* to justify these contradictory views beyond the arguments presented above.

Elkan's theorem says that if we assume $\neg(A\wedge\neg B) = B\vee(\neg A\wedge\neg B)$ for all sentences A and B in standard min/max fuzzy logic, then for any propositions P and Q it follows that the truth value of P is either the same as the truth value of Q or its negation. This means that if fuzzy logic would follow this classical equivalence it will be useless at best.

Our goal is to give an explanatory model to shed light on this issue. The model uses a concept of a *logic of uncertainty* of rational and irrational agents that includes both classical logic, fuzzy logic and other logics. We state that classical logic can model only rational agents, but fuzzy logic can model irrational agents too. Rational agents do not break logic equivalence, but irrational agents do this.

We resolve the controversy about equivalence by showing that classical logic and fuzzy logic have different domains of rational and irrational agents, respectively. It does not mean that there is no overlap, where both theories model rational agents, but this issue is beyond the scope of this paper.

2 Explanatory model

2.1 The scope of logic

To analyze the controversy we start from the basic question: "What is logic?' Obviously, there are many possible answers, but it is important for our analysis to stress that logic is a reasoning mechanism on *abstracted sentences*. Abstraction means that out of the infinite number of properties $\{p\}$ of sentence S in a natural language only few properties $\{p_{abs}\}$ are associated with an abstracted sentence S_A. Now we need to distinguish between *logic* and *model* of the real world situation. Logic can be viewed as a model of a real word expressed in abstracted sentences.

Models that use less expressive languages may produce less realistic model of the world. For instance, the propositional logic is much more limited than the first order logic (FOL). However, the classic logic has no goal to build a comprehensive model of a real world situation. By definition, the classic propositional logic tries to construct truth-values of some sentence from *only truth-values* of other sentences, not from semantic *contents* of sentences themselves. It seems not accidental that the probability theory is not called a logic. The probability theory is not limited by this assumption.

The classical logic uses a very limited semantics of sentences. Logics of uncertainty should use much more semantics to be able realistically model complex uncertain situations [12]. Involving much more semantic information implies that the situation and a logic that models the situation become interdependent. They have a high overlap in the set parameters. The question is "How to incorporate semantics to the logic of uncertainty?"

In this paper, we show that the use of a mixture of rational and irrational agents is a way to incorporate important semantic information.

2.2. Elkan's paradox and its meaning as a break of symmetry in logic

The proof of the Elkan's theorem [6] is based on the *assumption of classical equivalence applied to the fuzzy logic*. Specifically, it uses classic equivalence $\neg(A\wedge\neg B)= B\vee(\neg A\wedge\neg B)$, which does not hold in fuzzy logic with A and B from [0,1] and operations, $A \wedge B = \min(A,B)$, $B \vee A = \max(A, B)$, $\neg A =1- A$.

It raises the questions: "Should we require the classical logic equivalence of sentences for every logic of uncertainty? "What should logic of uncertainty be?" If the classical equivalence is not required and justified in fuzzy logic then Elkan's paradox is not a paradox. It is proposed in [2] that another relation, which is a kind of similarity relation, should substitute the classical equivalence in fuzzy logic. Given membership function μ with only two values {0,1} the classic logic holds properties:

(1) $\mu (X \wedge Y) = \mu (X) \cdot \mu (Y)$
(2) $\mu (X \vee Y) = \mu (X) + \mu (Y) - \mu (X) \mu (Y)$
(3) $\mu (\neg X) = 1 - \mu (X)$
(4) $\mu (X \wedge Y) = \mu (X) - \mu (\neg Y \wedge X) = \mu (Y) - \mu (\neg X \wedge Y)$
(5) $\mu (X \vee Y) = \mu (X) + \mu (Y \wedge \neg X) = \mu (Y) + \mu (X \wedge \neg Y)$
(6) $\mu (\neg X) = \mu (\neg X \oplus IR)$,

where operation "$\oplus$ " is XOR and $IR= \neg X \wedge X$. In classical logic, IR is always false, IR=0. Therefore, $\mu(\neg X)=1 \Rightarrow \mu(\neg X\oplus IR)=1$, and $\mu(\neg X)=0 \Rightarrow \mu(\neg X \oplus IR)=0$. Thus, in classical logic adding IR using XOR to any statement X keeps logic equivalence. Below we show that this is not case for the irrational agents that allow $\neg X \oplus IR$ to be true.

2.3 Reasoning with conflicting and irrational evaluations

As a departure from the classical logic, the proposed logic of uncertainty allows the proposition $\neg X\wedge X$ to be not false, $\mu(\neg X\wedge X) > 0$, in accordance with the evaluation provided by an irrational agent. If this happened, we call $\neg X\wedge X$ an irrational sentence. We will also call any other sentence as an *irrational sentence/statement* and denote it as IR if it takes a value that is impossible in the classical logic. For instance, we have $\mu (X\vee\neg X)=1$, that is a classical logic tautology, but an irrational agent may claim that $\mu (X\vee\neg X)<1$. In this case, $X\vee\neg X$ is an irrational sentence and can be denoted as IR, $X\vee\neg X=IR$. Accordingly, the agent that provides irrational sentences/statements is called an *irrational agent*.

In the classical logic, a conjunction of X with any false statement S, μ(S)=F, is false, μ(X∧S)=F. Thus, if the knowledge base (KB) of the domain contains such S then the reasoning in this KB is useless. It is an attempt in any useful KB to make it fully consistent (without contradiction that leads to the false statements in the KB). However, this is difficult to accomplish due to the inevitable incompleteness of descriptions of a real world object used in KBs. We may have two patients with the same description D in KB, but for one of the patients the statement S(D) is true but for another one S(D) is false, that is one of them has cancer and other one does not. Thus, S(D) is an irrational statement in our terms, we will write this as, S(D)=(T,F), that is assigning to S(D) a vector of values (T, F).

The concept of the IR statement allows us to build a logic of uncertainty that separates *reasoning with IR sentences* from reasoning without them. The reasoning without IR sentences can follow *classical reasoning rules,* while the reasoning with IR can be done differently.

2.4 Conflicting and irrational evaluation by a set of agents

In the proposed agent approach to the logic of uncertainty, each agent gives logic values to the statements including the irrational evaluations. Thus, we have a *vector of logic evaluations*, e.g.,

$$t(A) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ true & false & true & false \end{bmatrix} \tag{7}$$

In this example, agents together are *incoherent* for statement A because for two agents A is true and for two other agents A is false. We can associate a *vector evaluation t*(*A*) with a numeric *scalar evaluation*, commonly known as a membership value in fuzzy logic

$$\mu(A) = \frac{\mu(Agent_1, A) + ... + \mu(Agent_4, A)}{4} = \frac{1+0+1+0}{4} = \frac{2}{4}$$

Respectively operations ∧ and ∨ can be defined as follows

$$\mu(X \wedge Y) = \min(\mu(X), \mu(Y)) - \mu(\neg X \wedge Y) = \mu(Y) - \mu(\neg X \wedge Y)$$

$$\mu(X \vee Y) = \max(\mu(X), , \mu(Y)) + \mu(\neg X \wedge Y) = \mu(X) + \mu(\neg X \wedge Y)$$

Next, we define a statement S as an *irrational statement* (IR) if *two contradictory values* are given to S by the agent in some order, (TF) or (FT). In addition statements such as X ∧¬X =True and X ∨¬X = false can be called irrational statements, IR.

The agent G is in *irrational state* for statement S if S=IR for the agent G. We also use vector forms (8) and (9).

$$\begin{bmatrix} true \\ false \end{bmatrix}, \begin{bmatrix} false \\ true \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} true \\ true \end{bmatrix} = [true], \begin{bmatrix} false \\ false \end{bmatrix} = [false], \tag{9}$$

In (9) values are repeated therefore to shorten notation we will also use [true] and [false] as show in (9). Form (8) is applicable only to irrational agents (8) and form (9) is applicable both types of agents. Irrational agents can use form (9) saying that A∨¬A is always false for these agents,

$$t(\neg A \vee A) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & false & false & false \end{bmatrix} \tag{10}$$

In our explanatory model, the irrationality statement (10) is a result of A and ¬A evaluated by each agent in a vector form as irrational statements shown below.

Here every agent first evaluates A as $\begin{bmatrix} false \\ true \end{bmatrix}$ and then evaluates ¬A as $\begin{bmatrix} true \\ false \end{bmatrix}$,

$$t(A) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ \begin{bmatrix} false \\ true \end{bmatrix} & \begin{bmatrix} false \\ true \end{bmatrix} & \begin{bmatrix} false \\ true \end{bmatrix} & \begin{bmatrix} false \\ true \end{bmatrix} \end{bmatrix}$$

$$t(\neg A) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ \begin{bmatrix} true \\ false \end{bmatrix} & \begin{bmatrix} true \\ false \end{bmatrix} & \begin{bmatrix} true \\ false \end{bmatrix} & \begin{bmatrix} true \\ false \end{bmatrix} \end{bmatrix}$$

In this notation, we introduce a *matrix logic* that includes matrix evaluations and matrix operations. Using vector logic evaluations (8) and (9) we can generate a set of matrix evaluations that represent rationality/irrationality of a set of agents for IR = X∨¬X as it is shown in the example below for four agents. We remark that four states presented in (8) and (9) are possible for any agent.

Vector and matrix representations of logic were studied in several publications [4], [9], [10], [11]. In [9] a Boolean vector of logic evaluations is assigned to statement S and classical logic operations is applied to each coordinate of the vector evaluation. We expand further this representation to a matrix of Boolean evaluations of S,

$$t(S) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} & \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} & \begin{bmatrix} s_{13} \\ a_{23} \end{bmatrix} & \begin{bmatrix} s_{14} \\ s_{24} \end{bmatrix} \end{bmatrix}$$

Matrix Boolean operations are defined by applying classical logic operations to each cell individually. In (11) “ $\square$ ” means any binary logic operation, e.g., ∧,∨, ⊕.

$$t(S \square Q) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ \begin{bmatrix} s_{11} \square q_{11} \\ s_{21} \square q_{21} \end{bmatrix} & \begin{bmatrix} s_{12} \square q_{12} \\ s_{22} \square q_{22} \end{bmatrix} & \begin{bmatrix} s_{13} \square q_{13} \\ a_{23} \square q_{23} \end{bmatrix} & \begin{bmatrix} s_{14} \square q_{14} \\ s_{24} \square q_{24} \end{bmatrix} \end{bmatrix} \tag{11}$$

The negation $t(\neg S)$ is defined by negating each s_{ij} individually as we did in the previous section. This matrix logic can be expanded further from $2\times n$ matrixes to $m\times n$ matrixes by adding more rows.

3. Elkan's expressions and irrationality

First we *irrationalize* each side of ¬(A∧¬B) = B∨(¬A∧¬B) and then compare them showing that irrationalized side are not equivalent. We convert ¬B to ¬B⊕ IR, combine it with *A* producing A∧(¬B⊕IR). The we negate it, ¬(A∧(¬B⊕ IR)) and combine with IR to produce ¬(A∧(¬B⊕ IR)) ⊕IR that is an irrationalized form of ¬(A∧¬B). This process converts rational ¬B to irrational ¬B⊕ IR. The justification for such transformation is provided in section 2.2 with property (6), μ (¬X) = μ (¬X ⊕ IR), known for the classical logic if IR = (S ∧¬S).

3.1 First expression

Consider statement ¬B evaluated by four agents as shown in (12),

$$t(\neg B) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ true & false & true & true \end{bmatrix} \tag{12}$$

Next we convert $\neg B$ to $\neg B\oplus IR$ using $t(IR)$ given in (11)

$$t(\neg B \oplus IR) =$$

$$\begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ true \oplus false & false \oplus true & true \oplus false & true \oplus true \end{bmatrix} = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ true & true & true & false \end{bmatrix} \tag{13}$$

This allows us to write $t(A\wedge(\neg B\oplus IR))$ using $t(A)$ from (7)

$$t(A \wedge (\neg B \oplus IR)) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ true & false & true & false \end{bmatrix}$$

This is a base to produce $t(\neg(A\wedge(\neg B\oplus IR)) \oplus IR)$,

$$t((\neg(A \wedge (\neg B \oplus IR))) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & true & false & true \end{bmatrix}$$

$$t(\neg(A \wedge (\neg B \oplus IR)) \oplus IR) = \begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & false & false & false \end{bmatrix} \tag{14}$$

3.2 Second expression

The irrationalized version of $B\vee(\neg A\wedge\neg B)$ using XOR is

$$B\vee(\neg A\oplus IR)\wedge(\neg B\oplus IR)$$

with the same A and B that we used in the first expression and using (13) we get

$$t(\neg A)=\begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & frue & false & frue \end{bmatrix}$$

$$t(\neg A\oplus IR)=\begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & false & false & false \end{bmatrix}$$

$$t((\neg A\oplus IR)\wedge(\neg B\oplus IR))=\begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & false & false & false \end{bmatrix}$$

$$t((B\vee(\neg A\oplus IR)\wedge(\neg B\oplus IR)))=\begin{bmatrix} Agent_1 & Agent_2 & Agent_3 & Agent_4 \\ false & true & false & false \end{bmatrix} \quad (15)$$

The comparison of (14) and (15) shows that $\neg(A\wedge\neg B)\neq B\vee(\neg A\wedge\neg B)$ when we have irrational agents. It explains why classical equivalence is violated. We view fuzzy logic as a scalar version of this vector logic that uses formula (16).

$$\mu(S)=\frac{1}{n}\sum_{i}^{n}\mu(Agent_i,S) \quad (16)$$

The number of all true values in (14) is 0, but it is 1 in (15). Thus $\mu(\neg(A\wedge\neg B))=0$, and $\mu(B\vee(\neg A\neg B)=1/4$ are different for classically equivalent statements. In this way, the presented vector and matrix logic provides an explanatory model for the Elkan's paradox.

5 Conclusion

This paper presented a logic of irrational agents. The semantics of situation that is modeled by this logic is a much more complex that is modeled by the scalar classical logic with its equivalence. We defined vector/matrix representation and logic operations for sets of agents. This provides a new way to study the uncertainty phenomena. The matrix and vector logic approach is a new instrument to study the irrationality and inconsistency of agents. Agents are modeled as a group by using a special type of logic evaluations - matrixes. In this way, we change the individual scalar classical true and false values to a global or vector definition of True and False for a set of agents. The proposed explanatory model resolves the Elkan's paradox. We had shown that for irrational agents the Elkan's paradox is not a paradox but the explained phenomenon. This is applicable to fuzzy logic interpreted as a scalar logic of a mixture of agents that includes self-conflicting and irrational agents. If the agent that uses only T/F evaluations is consistent and rational (has no self-conflict and does not evaluate tautology $x \vee \neg x$ as false and contradiction $x \wedge \neg x$ as true) the logic equivalence for such agents should be the same as in the classical logic. If such consistent and rational agents use evaluation values from the whole interval [0,1], then we do not see a reason why classical equivalence should be abandoned and broken. This can be a subject of further exploration.

In this paper, we formalized *multiple evaluations* in the vector/matrix logic while the multi-valued logics formalize a *single scalar evaluation* that has three or more values in [0,1] interval in contrast with classical logic.

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