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Agents' model of uncertainty

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Abstract Multi-agent systems play an increasing role in sensor networks, software engineering, web design, e-commerce, robotics, and many others areas. Uncertainty is a fundamental property of these areas. Agent-based systems use probabilistic and other uncertainty models developed earlier without explicit consideration of agents. This paper explores the impact of agents on uncertainty models and theories. We compare two methods of introducing agents to uncertainty theories and propose a new theory called the agent-based uncertainty theory (AUT). We show advantages of AUT for advancing multi-agent systems and for solving an internal fundamental question of uncertainty theories, that is identifying coherent approaches to uncertainty. The advantages of AUT are that it provides a uniform agent-based representation and an operational empirical interpretation for several uncertainty theories such as rough set theory, fuzzy sets theory, evidence theory, and probability theory. We show also that the introduction of agents to intuitionist uncertainty formalisms can reduce their conceptual complexity. To build such uniformity the AUT exploits the fact that agents as independent entities can give conflicting evaluations of the same attribute. The AUT is based on complex aggregations of crisp (non-fuzzy) conflicting judgments of agents. The generality of AUT is derived from the logical classification of types (orders) of conflicts in the agent populations. At the first order of conflict, the two agent populations are disjoint and there is no interference of logic values assigned to any statement p and its negation by agents. The second order of conflict models superposition (interference) of logic values for overlapping agent populations where an agent assigns conflicting logic values (true, false) to the same attribute simultaneously.

Keywords Uncertainty logic · Conflicting agents · Ignorance · Contradiction · Fuzzy logic · Evidence theory · Intuitionistic fuzzy sets · Rough sets

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1 Introduction

In this paper, we explore the uncertainty that is caused by contradiction between agents and by self-conflict of individual agents. Self-conflicting agents can be called irrational agents in contrast with rational agents that have no self-conflict. The concept of rational agents is explored in several fields. In [28], it is motivated by fundamental goals to provide a formal logicization of software engineering [6]. The rationality of a software agent rests in the maximization of its chances of success based on software agent's knowledge of its environment. Any agent that maximizes chances of success without any logical conflict is considered as a rational one. In fact, an agent's success criterion as well as the knowledge of the environment can be quite conflicting from the logic viewpoint. In this situation, the agent can be in a logical state that we can define as irrational. Accordingly, agent's behavior is not rational.

The research area that is deeply interested in partially rational agents called boundedly rational agents is motivated by fundamental goals in psychology and economics [10,15]. The works in this area explore the psychology of intuitive beliefs and choices by examining their bounded rationality as Kahneman outlined in his Nobel Prize Lecture in 2002. In economics, bounded rationality means use of (1) multiple utility functions instead of one global scalar utility function, (2) limited types of utility functions due to cost of information collection, and (3) heuristics.

Our concept of the agent in a conflicting logical state (irrational state) is motivated by fundamental goals in the artificial intelligence area of logic and reasoning under uncertainty [12,13] with agents [8]. We envision a formal logicization of reasoning of irrational agents that do not follow rational reasoning in the frames of the classical logic and the probability theory and even fuzzy logic. This area is quite different from two previous areas as well as the areas that study expert reasoning simply because irrational agents hardly fit a definition of experts.

As stated in [14] the language for describing inconsistency is underdeveloped for many applications. This is related to the general problem of agent communication language [18].

It is difficult to qualify inconsistency and to say which agent's database (a set of formulas) is more inconsistent than another one. A general characterization of inconsistency as the number of minimally inconsistent subsets of formulas that would be eliminated in used in [14]. This technique is considered as useful for monitoring progress in agents' negotiations and in agents' comparing heterogeneous sources of information.

We define a specific concept of an ME-rational (the agent that is rational relative to mutual exclusion) with a clear criterion how to distinguish such agents. Similarly, we define a concept of an ME-irrational agent as a self-conflicting agent as well as contradictory agents that contradict each other but without self-conflict in judgment. The concepts of conflict and self-conflict in judgments are critical for our study of irrational agents, because uncertainty often means that several contradictory statements are made. The concept of self-conflict itself is studied in paraconsistent logics [24] and multi-objective optimization. We build our approach on the results of direct measurements or answers of agents not on the aggregated utility functions to be able to model a structure of contradictions explicitly.

The probability calculus does not incorporate explicitly the concepts of irrationality or agent state of logic conflict. It misses the structural information at the level of individual objects, but preserves the information on a global property of a set of objects. Given a dice the probability theory studies frequencies of the different faces $E = \{e\}$ as independent elements. This set of elementary events E has no structure. It is only required that elements of E are mutually exclusive and complete, that is no other alternative is possible. The order of its elements is irrelevant to probabilities of each number in E. No irrationality or conflict is allowed in this definition. The classical probability calculus does not provide a mechanism for modelling uncertainty when agents communicate (collaborates or conflict). Recent work by Halpern [12] is an important attempt to fill this gap.

This paper introduces a hierarchy of logics of uncertainty in the context of conflicting evaluations of preferences by agents that includes a mechanism for identifying a type of logic and reasoning computations in the agent logic. We call this hierarchy an agent-based uncertainty theory (AUT). It is a further development of our previous works [19–22,25,26] that contain extensive references to related work. The fundamental analysis and review of relevant issues can be found in [3,6–8].

There are two basic ways to assign initial uncertainty values by agents. The first way is to ask the agent to evaluate a classical truth-value logic function $v(p) \in \{\text{False}, \text{True}\}$ of a proposition p and then to compute frequencies of these answers provided by a set of agents G. The second way is to ask each agent to give a degree of truth directly as a truth-value logic function $v(p)$ in the [0,1] interval.

This direct degree can be interpreted as a probability of the statement p [3,12,13] or as a value of the fuzzy logic membership function (MF) [13]. The status of MF is a subject of extensive discussion. It was interpreted probabilistically [11,13,20,21]. There are also extensive literature that states that MFs represent a type of uncertainty that differs from probabilistic one [2,4,5,16,17]. In other words, this question is equivalent to the question: "Is probability the only coherent approach to uncertainty?" [4]. The answers in the literature range from "Yes" to the opposite extreme that accepts any ad hoc uncertainty model based on t-norms and t-co-norms. One of the goals of this paper is to build a conceptual base to resolve this MF interpretation issue.

This paper is organized as follows. Section 2 describes base concepts and definitions. Section 3 contains a framework of first order of conflict logic state. Section 4 describes the order of logic conflicts in terms of self-conflict and irrationality. Section 5 concludes the paper.

2 Concepts and definitions

Two common ways to introduce agents to the uncertainty theory differ in the mode of getting initial uncertainty values from the agents. In the first one each agent makes a crisp evaluation of the proposition (T, F) and then frequencies are computed for a set of agents G. In the second one, the agent directly assigns an uncertainty value (membership value, probability, etc.). At first glance, the second method is very intuitive, but in fact, it leaves the open question how to make and justify a coherent resonating and logic operations with such direct values.

A probability value can be agent's judgment or a result of physical experiments. In [7] agents are connected to logic and probability in the following way. In a given state s, the formula $P_j(\varphi)$ denotes the probability of the logic proposition φ according to the agent j's probability distribution in the state s. In this model, an individual agent gives the probability. All agents are autonomous and no conflict among agents and self-conflict is modeled explicitly. This formalization does not provide tools to fuse the contradictory knowledge of different agents.

A reasoning agent g is called totally consistent for statement S if g always provides the same truth value for S, thus the change of context C and time t does not change S value for agent g. Note that total consistence does not imply correctness of the evaluation of S by g

because a consistent truth-value can be incorrect. Such agents and their reasoning are modeled in the classical logic. A reasoning agent g is called partially consistent for statement S if g not always provides the same truth-value for S. If sources of inconsistency are numerous, cannot be traced or cannot be explicitly presented then the probability theory is used commonly to model such agents and their reasoning with probability $P(S) = n/N$, where n is the number of cases when g answered that S is true and N is a total number of cases. Otherwise, the classical logic still can be used.

Similarly, a reasoning agent g is called totally consistent for a set of statements $\{S\}$ if g always provides the same truth value for each S from $\{S\}$. Again, the classical logic is applicable for such agent. A reasoning agent g is called partially consistent for a set of statement $\{S\}$ if g not always provides the same truth-value for some or all statements from $\{S\}$. The expansion from a single statement S to a set of statements $\{S\}$ can be done differently with different relations among statements in the $\{S\}$ for the agent g. In addition, set $\{S\}$ may contain statements about more than one object, event. Relations among statements and objects define applicability of the probability theory. Kolmogorov's axioms of the probability theory set up these relations.

One of the most important of these axioms is a mutual exclusion axiom for elementary events. Let E be a predicate which is true if e is an elementary event, $\{e\}$ be a set of all given elementary events, and $P(e)$ be a probability of event e, then $P(e_i \wedge e_j) = 0$ and $P(e_i \vee e_j) = P(e_i) + P(e_j)$. Property $P(e_i \wedge e_j) = 0$ means that e_i and e_j cannot happen simultaneously. Let $S(e) =$True if elementary event e has happened. If agent g tells that e_i has happened, $S(e_i) =$ True, then g should tell $S(e_j) =$ False, i.e., $S(e_i) \wedge S(e_j) =$ False for g. We will call this agent g rational on Mutual Exclusion (ME-rational agent). Accordingly we will call agent g irrational on Mutual Exclusion (ME-irrational agent) if g tells both $S(e_i) =$ True and $S(e_j) =$ True for mutually exclusive events e_i and e_j, $S(e_i) \wedge S(e_i) =$ True. In other words, we can say that for this agent g contradiction $S(e_i) \wedge S(e_j)$ is True and the agent is in a conflicting state.

Each individual agent can be very consistent, but agent g_1 can contradict agent g_2 creating a pair of conflicting agents (g_1, g_2). We distinguish between a state of logic conflict among agents, a state of logic conflict inside the agent. We call later agents irrational agents. The set of such irrational agents is in the conflicting logic state.

A self-conflicting agent g is in a synchronic state of logic conflict where logic values true and false interfere one with the other in a complex way. A self-conflicting agent can provide two or more contradictory preferences, $(p, \neg p)$ with $p = (A > B)$ and, $\neg p = (B > A)$ for statement p by stating that p is true and false at the same time. This can be an indication that g has two or more different competing criteria, $C_1, C_2, \ldots, C_N$, e.g., C_1 minimizes price and C_2 maximizes quality. However, this does not resolve the ultimate preference contradiction it only explains it because the goal is to make a preference between A and B and the explanation does not solve this problem.

If $C_1, C_2, \ldots, C_N$ criteria are in fact the same criterion C, $\forall i C_i = C$, then it is possible to have a set of agents in conflicting logic states relative to C, but agents are not in a state of logical self-conflict because every agent produces only one logic value using C. Thus, a set of agents in the state of logic conflict is a subset of the class of self-conflicting agents. The agent can be in a logic conflict state because of inability to understand complex context and evaluate criteria. In the stock market, some traders quite often do not understand a complex context and rational criteria of trading. These traders can be in a logic conflict exhibiting chaotic, random, and impulsive behavior. At the same market situation, they can sell and buy stocks, exhibiting logic conflicting states "sell" $= p$ and "buy" $= \neg p$ in the same market situation that appear as irrational behavior with $p \wedge \neg p$ is not always false.

A logical structure of self-conflicting states and agents is much more complex than without self-conflict. In the case of N binary criteria $C_1, C_2, \ldots, C_N$ which evaluate the logic value for the same attribute, there are 2^N possible states and $2^N - 2$ of them can exhibit conflict between criteria. There is only two vector of values of the criteria that have no contradiction for sure, $(T, T, \ldots, T)$ and $(F, F, \ldots, F)$. Transitions among the states are possible as Montero [23] shows with the linguistic definition of the logic states such as Ignorance I, Contradiction C, Poor P, Good G, and others. In this situation, a dynamic logic system can be built that represents the dynamic change of the logically self-conflicting states for different logics of uncertainty.

3 Framework of first order of conflict logic state

To define a concept of first order conflicting agents we set up a framework and use an example of agents who buy cars. The concept of first order conflicting agents is much wider than provided in this example with a binary preference relation. Using this preference relation we show that at the first order of conflicts AND and OR operations should not be traditional classical logic operations, but operations on a vector space of the agents and aggregation to reflect a conflicting structure of self coherent population of agents evaluations.

Consider 100 agents g_i, two cars A and B and preference relation or criteria C = "$>$" between cars to be assigned by each of these agents (potential buyers).

We define a Boolean variable X such that $X = 1$ (True) if $A > B$ else $X = 0$ (False). Each agent g_i answered a questionnaire with two options offered for the criteria C:

(1) "$A > B$ is true" and (2) "$A > B$ is false".

Say 70 agents marked "$A > B$ is true" giving a frequency value, $m(A > B) = 70/100$, that can be interpreted as a membership function value or a measure of the conflict logic state for the universal set of the 100 of agents.

The situation for which a group of agents assumes that $A > B$ is true and a complementary group of agents assumes that the same $A > B$ is false, is a situation of conflict/contradiction among agents that are wrestling for the logic value of $A > B$. We denote this situation as a first order of conflict logic state. Here each individual agent is completely rational and has no self-conflict. The conflict exists only between different agents when they evaluate the same statement with the criteria C.

We specify a set of agents G, a subset of agents $G(A > B)$ in a logic state for which $A > B$ is true and a set of agents $G(A < B)$ for which $A > B$ is false,

$$G(A > B) = \{g \in G | A > B \text{ is true}\}, \quad G(A < B) = \{g \in G | A > B \text{ is false}\}.$$

Thus, we have $G(A > B) \cap G(A < B) = \emptyset$.

Example Let $G = \{\text{Agent}_1, \text{Agent}_2, \text{Agent}_3, \text{Agent}_4\}$ and $G(A > B) = \{\text{Agent}_1, \text{Agent}_4\}$ and $G(A < B) = \{\text{Agent}_2, \text{Agent}_3\}$. The set G is in a conflict logic state. The evaluation for all four agents and the criteria C = "$>$" can be recorded in this vector form,

$$\text{value}(A > B) = \begin{bmatrix} \text{Agent}_1 & \text{Agent}_2 & \text{Agent}_3 & \text{Agent}_4 \\ \text{True} & \text{False} & \text{False} & \text{True} \end{bmatrix}.$$

Similarly, for N agents we can record the evaluations as follows

$$V(A>B)=\begin{bmatrix} \text{agent}_1 & \text{agent}_2 & \dots & \text{agent}_N \\ v_1(A>B) & v_2(A>B) & \dots & v_N(A>B) \end{bmatrix}.$$

In a graphic way, this can be represented in the agent space as shown in Fig. 1.

Below we show that at the first order of conflicts, AND and OR operations should not be the traditional scalar classical logic operations, but should be vector operations in the space of the agents that can be aggregated to a scalar value in [0,1] interval later if needed, but aggregation may bring loss of the information. The vector operations reflect a structure of logic conflict among coherent individual agent evaluations.

In fact, at the first order of agent conflict we must introduce a fusion process of the logic values given by the agents. A basic way to do this is to compute the weighted frequency of preferences of all agents for the sentence p that can be equal to $A > B$.

$$\mu(p)=\frac{w_1v_1(p)+\cdots+w_Nv_N(p)}{N}=\frac{1}{N}\begin{bmatrix} v_1(p) \\ v_2(p) \\ \dots \\ v_N(p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{bmatrix} \tag{1}$$

where $\begin{bmatrix} v_1(p) \\ v_2(p) \\ \dots \\ v_N(p) \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{bmatrix}$ are two vectors in the space of the agents' evaluations. The first vector contains all logic states (true/false) for all the agents, the second vector (with property $\sum_{k=1}^{N} w_k = 1$) contains non-negative weights (utilities) that are given to each agent in the fusion process. In a simple frequency case, each weight is equal to $1/N$.

At first glance, $\mu(A > B)$ is the same as used in the probability and utility theories. However, classical axioms of the probability theory have no references to agents producing initial uncertainty values and do not violate the mutual exclusion. As a result, formulas for assigning initial uncertainty values such as $\mu(A > B)$ are not a part of the theory. Value $\mu(A > B)$ can differ significantly from the classical relative frequency for some irrational agents.

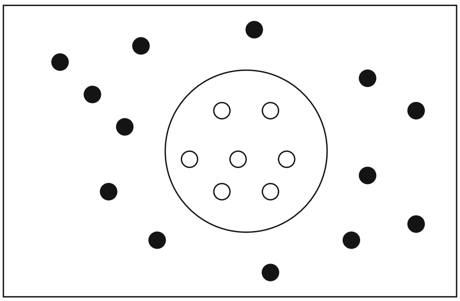


Fig. 1 The set of 20 agents in the logic conflicting state, where 7 *white agents* are in the state true and 13 *black agents* are in the state false for the same preference criteria $C =$ "$>$"

In the first order of conflict, logic states of the agent space vector logic operations are defined as

$$V(p \wedge q) = \begin{bmatrix} \text{agent}_1 & \dots & \dots & \text{agent}_N \\ v_1(p) \wedge v_1(q) & \dots & \dots & v_N(p) \wedge v_N(q) \end{bmatrix},$$

$$V(p \vee q) = \begin{bmatrix} \text{agent}_1 & \dots & \dots & \text{agent}_N \\ v_1(p) \vee v_1(q) & \dots & \dots & v_N(p) \vee v_N(q) \end{bmatrix},$$

$$V(\neg p) = \begin{bmatrix} \text{agent}_1 & \dots & \dots & \text{agent}_N \\ \neg v_1(p) & \dots & \dots & \neg v_N(p) \end{bmatrix},$$

where the symbols $\wedge$, $\vee$, $\neg$ in the right side of the equations are the classical AND, OR, and NOT operations. For a set of conflicting agents at the first order of conflicts, we have these important properties

$$\begin{aligned} |G(p \wedge q)| &= \min(|G(p)|, |G(q)|) - |G(\neg p \wedge q)|, \\ |G(p \vee q)| &= \max(|G(p)|, |G(q)|) + |G(\neg p \wedge q)|, \end{aligned}$$

where $|G(q)| \leq |G(p)|$. The sets $|G(p)|$ and $|G(q)|$ are the set of respective agents for which statement p and q are true, respectively. If $G(q) \subset G(p)$ then we have well known min, max operations. We remark also that

$$G(\neg p) = G^C(p),$$

where $G^C(p)$ is the complementary set of $G(p)$ in G. Thus,

$$\begin{aligned} G(p \vee \neg p) &= G(p) \cup G(\neg p) = G(p) \cup G(p)^C = G \\ G(p \wedge \neg p) &= G(p) \cap G(\neg p) = G(p) \cap G(p)^C = \emptyset \end{aligned}$$

In other words, $G(p \wedge \neg p) = \emptyset$ corresponds to the contradiction $p \wedge \neg p$ that is always false and $G(p \vee \neg p) = G$ corresponds to the tautology $p \vee \neg p$ that is always true in the first order conflict.

The symmetric difference $A \oplus B$ is defined as $(A \cap B^c) \cup (A^c \cap B)$. In other words, if p is true in A and q is true in B we can write the previous set expression in this way for

$$A \oplus B = G((p \wedge \neg q) \vee (\neg p \wedge q)).$$

In agent terms, $A \oplus B$ means a set of agents for which A p is true in A or q is true in B, but not together p and q. We can define

$$D(A, B) = |A \oplus B| = |G((\neg p \wedge q) \vee (p \wedge \neg q))|$$

as the number of elements in the set $A \oplus B$ and notice that $D(A, B)$ a is a distance, i.e., $D(A, B)$ satisfies all three axioms of the distance [9],

$$\begin{aligned} &D(a, b) \geq 0 \\ &D(a, b) = D(b, a) \\ &D(a, b) + D(b, c) \geq D(a, c). \end{aligned}$$

Figure 2 shows a set of agents for which p is true and a set of agents for which q is true. The number of agents for which $p \wedge q$ is true is connected with the distance $D(A, B) = |G(A \vee B)| = |G((\neg p \wedge q) \vee (p \wedge \neg q))|$, because if the number of agents for which $p \wedge q$ is true is maximum then the distance $D(A, B)$ reaches its minimum. If a set, where $p \wedge q$ is true is empty then the distance $D(A, B)$ reaches its maximum.

In Fig. 2a, set 2 with property $(p \wedge \neg q)$ consists of two agents $|G(p \wedge \neg q)| = 2$ and set 4 with property $(\neg p \wedge q)$ is empty, $|G(\neg p \wedge \neg q)| = 0$, thus $D(A, B) = 2$. The emptiness for $(\neg p \wedge q)$ means that a set of agents with true p includes a set of agents with true q.

In Fig. 2b, set 2 with property $(p \wedge \neg q)$ consists of four agents $|G(p \wedge \neg q)| = 4$ and set 4 with property $(\neg p \wedge q)$ consists of two agents, $|G(\neg p \wedge \neg q)| = 2$, thus $D(A, B) = 6$.

If one of the sets $G(p)$ and $G(q)$ includes each other (case a in Fig. 2) then

$$|G(p \wedge q)| = |G(p) \cap G(q)| = \min(|G(p)|, |G(q)|) = 3 = |G(q)|$$
$$|G(p \vee q)| = |G(p) \cup G(q)| = \max(|G(p)|, |G(q)|) = 5 = |G(p)|.$$

In this case nothing is added or subtracted in

$$|G(p \wedge q)| = \min(|G(p)|, |G(q)|) - |G(\neg p \wedge q)|,$$
$$|G(p \vee q)| = \max(|G(p)|, |G(q)|) + |G(\neg p \wedge q)|.$$

Therefore, the AND operation reaches its maximum value and the OR operation its minimum value. In the case b in Fig. 2 the situation is different with $D(A, B) = 6$, $G(\neg p \wedge q) = 2$ and

$$|G(p \wedge q)| = |G(p) \cap G(q)| = \min(|G(p)|, |G(q)|) - |G(\neg p \wedge q)| = 1$$
$$|G(p \vee q)| = |G(p) \cup G(q)| = \max(|G(p)|, |G(q)|) + |G(\neg p \wedge q)| = 7.$$

If sets of agents do not overlap then the distance between them reaches their maximum. Similarly, in such case the AND operation reaches its minimum and the OR operation reaches its maximum,

$$|G(p \wedge q)| = |G(p) \cap G(q)| = \min(|G(p)|, |G(q)|) - |G(\neg p \wedge q)| = 0$$
$$|G(p \vee q)| = |G(p) \cup G(q)| = \max(|G(p)|, |G(q)|) + |G(\neg p \wedge q)| = |G(p)| + |G(q)|.$$

Therefore, with the same $|G(p)|$ and $|G(q)|$ we may have different distances and different results for the AND and OR operations among the sets. These operations depend on internal structures of the agents' states, they are not simple truth-functional operations of classical

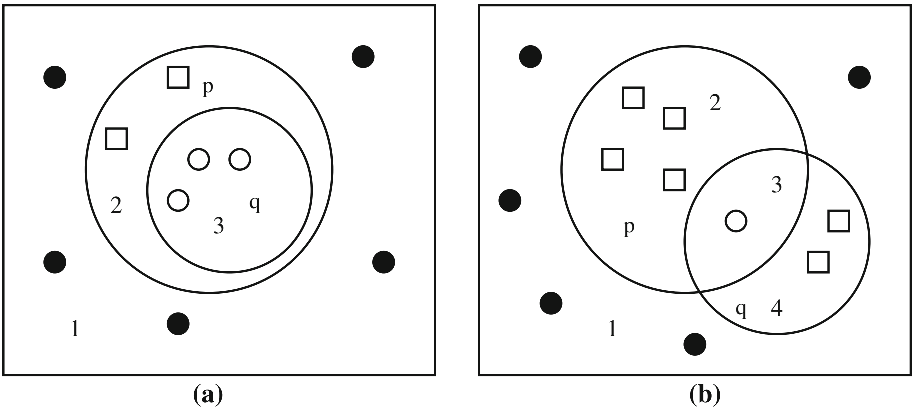


Fig. 2 The set of ten agents that form separate sets. Set *1* (*black circles*) consists of agents for which both p and q are false. Set *2* (*white squares*) consists of agents for which p is true but q is false. Set *3* (*white circles*) consists of agents for which p and q are true. Set *4* (*white squares* in **b**) consists of agents for which p is false and q is true

propositional logic. We remark that also the t-norm and t-conorm (that are truth-functional) can be redefined to become context dependent.

When we use the NOT operator we have in the classical logic that

$$G[\neg(p \wedge q)] = G[\neg p \vee \neg q] = G(\neg p) \cup G(\neg q) = \max(G(\neg p), G(\neg q)) + G(p \wedge \neg q)$$
$$G[\neg(p \vee q)] = G[\neg p \wedge \neg q] = G(\neg p) \cap G(\neg q) = \min(G(\neg p), G(\neg q)) - G(\neg p \wedge q).$$

For a complex AND, OR, NOT expressions we must define the graph of the distances among the sentences $p_1, p_2, \ldots, p_N$. The graph can be represented by the G symmetric matrix,

$$D = \begin{bmatrix} 0 & D_{1,2} & \ldots & D_{1,N} \\ D_{1,2} & 0 & \ldots & D_{2,N} \\ \ldots & \ldots & \ldots & \ldots \\ D_{1,N} & D_{2,N} & \ldots & 0 \end{bmatrix}.$$

For example, for three sentences p_1, p_2, and p_3, we have a graph of the distances shown in Fig. 3.

We also have a matrix $D = \begin{bmatrix} 0 & D_{1,2} & D_{1,3} \\ D_{1,2} & 0 & D_{2,3} \\ D_{1,3} & D_{2,3} & 0 \end{bmatrix}$ for the logic expression $h = p_1 \wedge p_2 \wedge p_3$. It can be represented as $h = (p_1 \wedge p_2) \wedge p_3$. Let $g = (p_1 \wedge p_2)$, then $h = g \wedge p_3$ and the distance between g and p_3 can be computed.

In this section, we formalized the concept of first order conflicting agents with the conflict only between different agents while each agent has no self-conflict. It is shown that for these conflicts the AND and OR operations should be defined as vector operations in the agent space. Such operations preserve the individual coherence of agent evaluations. Next, we had shown that evaluations for the first order conflicting agents can differ from traditional evaluations that do not consider conflict among agents.

4 Order of logic conflicts, self-conflict, and irrationality

At the first order of conflict, we have for the negation

$$G(\neg p) = G^C(p),$$

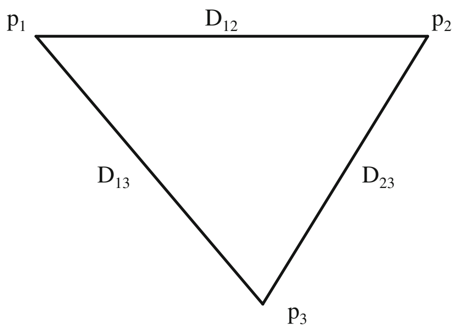


Fig. 3 Graph of the distances for three sentences p_1, p_2, and p_3

where $G^C(p)$ is a compliment to a set of agents $G(p)$ and can define

$$\mu(\neg p) = \frac{w_1 v_1(\neg p) + \cdots + w_N v_N(\neg p)}{N}$$
$$= \frac{1}{N}\begin{bmatrix} v_1(\neg p) \\ v_2(\neg p) \\ \cdots \\ v_N(\neg p) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_N \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 - v_1(p) \\ 1 - v_2(p) \\ \cdots \\ 1 - v_N(p) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_N \end{bmatrix} = 1 - \mu(p)$$

that is consistent with the known Zadeh's relation

$$\mu(\neg p) + \mu(p) = 1$$

Atanassov assumes in his model [1] the three categories: property p, its negation $\neg p$, and an indeterminacy s (about p or its negation) with the following definition of their relations,

$$\mu(\neg p) + \mu(p) + \mu(s) = 1.$$

This is different from Lukasiewicz's three-value logic with a fixed $\mu(s) = 0.5$ that is interpreted as possibility [23]. Atanassov's model is also a fuzzy partition as defined by Ruspini. Also the simple observation suggests a potential conflict between Zadeh's model Atanassov's model noted in [23]. To give a meaning to Atanassov's model we introduce the second order of conflict or the logical self-conflict (LSC) for agents.

We remark that at the first order of conflict the distance between $G(p)$ and $G(\neg p)$ is always equal to its maximum. This follows from the following analysis.

Let $A \oplus B = G((p \wedge \neg q) \vee (\neg p \wedge q))$ then for $q = \neg p$ we have

$$\begin{aligned} A \oplus B &= G((p \wedge \neg(\neg p)) \vee (\neg p \wedge \neg p)) \\ &= G((p \wedge p)) \vee (\neg p \wedge \neg p)) = G(p \vee \neg p) = \text{Universe}, \end{aligned}$$

that corresponds to the maximum of the distance.

This is caused by the use of only one criterion C to obtain the logic state (true or false) of the agent. In contrast, when we use two criteria C_1, C_2 to obtain the logic state of the agent, we can get more complex situations. In fact, we can get these logic states:

$$\begin{bmatrix} \text{state/criteria} & C_1 & C_2 \\ \text{State}_1 & \text{true} & \text{true} \\ \text{State}_2 & \text{true} & \text{false} \\ \text{State}_3 & \text{false} & \text{true} \\ \text{State}_4 & \text{false} & \text{false} \end{bmatrix}.$$

The states 1 and 4 are coherent states where the agent is not in a self-conflicting situation. The two criteria give the same logic value. The states 2 and 3 are self-conflicting logic states with two conflicting criteria one with the other to establish the logic state of the agent for the same attribute. Therefore, for the two criteria the universe of the agents consists of four classes. An example is shown in Fig. 4 where in the presence of two criteria C_1 and C_2, we have a particular type of negation set for p. In general, $G(\neg p)$ can contain any combination of (f, f), (f, t), and (t, f) types of agents from the universe of the agents. In the example in Fig. 4, (f, t) agents are not in $G(\neg p)$. These agents correspond to Atanassov's and Ruspini's models [23,27] with indeterminacy s:

$$G(p) \cup G(\neg p) \cup G(s) = \text{Universe}.$$

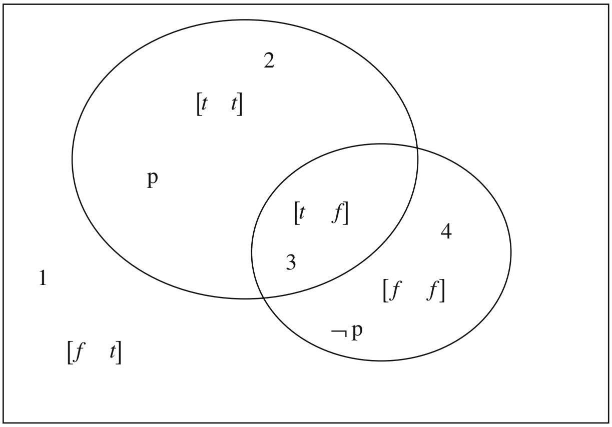


Fig. 4 Four classes of agents. The logic value true for p is located in (*2*, *3*), the negation of p is located in (*3*, *4*). Set (*3*) and set (*1*) are the logically self-conflicting states

Thus, when we have the fuzzy set "tall" we cannot compute the fuzzy set "not tall" without the knowledge of the distance between the fuzzy set "tall" and the fuzzy set "not tall" that we can define as "short".

The set (2,3) is "Tall", the set (4,3) is "short", which is different from the word "not tall" that is (1,4) this is in line with Atanassov's work. Now with our distance we can give the distance between tall set (3,2) and "short" (4,3). This is a fundamental novelty that goes beyond the Atanassov's theory.

At the first order, the distance between p and $\neg p$ is always the maximum. But at the second order, the distance between p and $\neg p$ is not always the maximum. In general it is less than the maximum. It is the minimum distance in Zadeh min rule. Thus, the definition of the distance is the same, but in the presence of the false tautology or true contradiction, we have many different distances among p and $\neg p$.

In set 3, we have the superposition of the negation $\neg p$ and p. In addition, in set 3, the negation enters to the set of agents where p is true. Thus, we have

$$G(p \wedge \neg p) = G(p) \cap G(\neg p) = G(3) \neq \emptyset.$$

Therefore, the contradiction is not empty. For the set 1 we have

$$\begin{aligned} G(p \vee \neg p) = G(p) \cup G(\neg p) = G(2, 3, 4) \subseteq \text{Universe} \\ G(1) = G(2, 3, 4)^C. \end{aligned}$$

Thus, the tautology does not cover all sets of agents. We remark that in the classical logic, if criterion C_1 is true, then also p is true, and if criterion C_2 is false then $\neg p$ is true. If $C_1 \equiv C_2$ and C_1 is true, then p is true and also C_2 is true. Also, if C_2 is false, then $\neg\ p$ is true and C_1 is false. Therefore, C_1 and C_2 cannot be distinguished at the logic level. In this case, the logical self-conflict is impossible, the contradiction and the tautology are classical.

However, if $C_1 \neq C_2$ then we may have an ignorance state which leads to the situation where tautology is not equal to 1. We can be unaware about presence of two different criteria, C_1 and C_2. Thus, we can have a contradiction state, that is a situation with contradiction that not false. Different aspects of concepts of ignorance or contradiction states are discussed in [23].

The sets in Fig. 4 can be obtained by the superposition of the two criteria C_1, C_2 evaluations as we show in Fig. 5.

Using a concept of the second order of conflict we can explain the contradiction of Zadeh's rule to the classical logic

$$\mu(p \wedge \neg p) = \min(\mu(p), \mu(\neg p)) = \min(\mu(p), 1 - \mu(p)) \geq 0.$$

The contradiction to the classical logic is that $\mu(p \wedge \neg p) > 0$ if $\mu(p) > 0$.

In fact in the agent approach to uncertainty, the rule of min for AND operation is valid only when $G(\neg p) \subseteq G(p)$. But this would contradict to $G(\neg p) \cup G(p)$ = Universe. To solve this problem we fixed the number of agents in $G(p)$ and $G(\neg p)$ thus

$$G(\neg p) \cup G(p) = \text{Universe}.$$

We assume also that $G(\neg p) \subseteq G(p)$. Thus, all agents in $G(\neg p)$ are in a state of logical self-conflict. We have also that $G(p)^C = \emptyset$, $G(p)$ is equal to the universe of the agents. With the introduction of the logically self-conflicting agents or contradictory agents, we solve the Zadeh's incoherence among the conjunction and the negation operations.

In Fig. 6, we have $G(p) \cup G(\neg p) = G(p) =$ Universe and $G(p \wedge \neg p) = \min(G(p),$ $G(\neg p)) = G(\neg p) > 0$ with $G(p)$ the universe of the agents and $G(\neg p) \subseteq G(p)$.

For the Sugeno negation,

$$\mu(\neg p) = \frac{1 - \mu(p)}{1 + \lambda\mu(p)} \quad \text{where } \lambda = [-1, \infty]$$

we have

$$\mu(\neg p) + \mu(p) = \frac{1 - \mu(p)}{1 + \lambda\mu(p)} + \mu(p)$$

and

$$\mu(\neg p) + \mu(p) - 1 = \lambda\mu(p)\left(\frac{\mu(p) - 1}{1 + \lambda\mu(p)}\right).$$

Therefore,

$$\begin{array}{ll} \text{if } \lambda \geq 0 & \text{then } \mu(\neg p) + \mu(p) \leq 1 \\ \text{if } \lambda < 0 & \text{then } \mu(\neg p) + \mu(p) > 1. \end{array}$$

Figure 7 shows the sets of agents for two cases of the parameter λ.

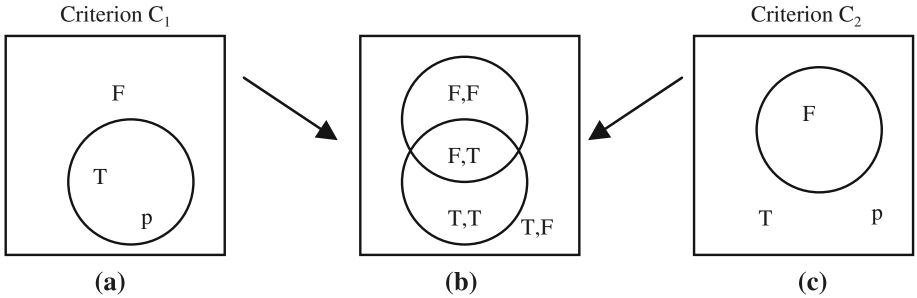


Fig. 5 Criteria and logic values in the second order of conflict. **a** Shows a set where p is true for the first criterion, **c** shows a set where p is true for the second criterion. **b** Shows the relations between two criteria. These criteria are not equal, the set where p is true for the first criterion is not equal for the second one. This generates the logically self-conflicting states for the agents shown in **b** with the ignorance and the contradiction

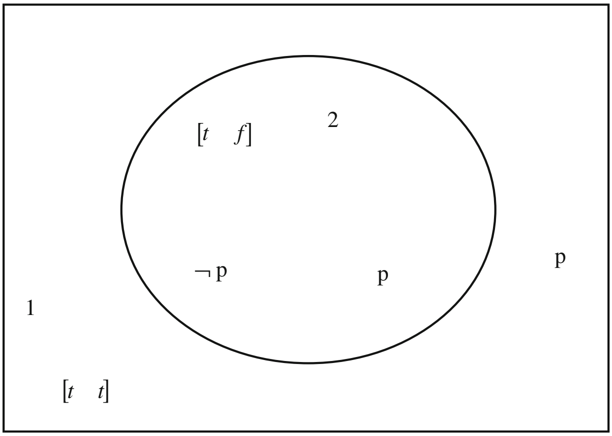
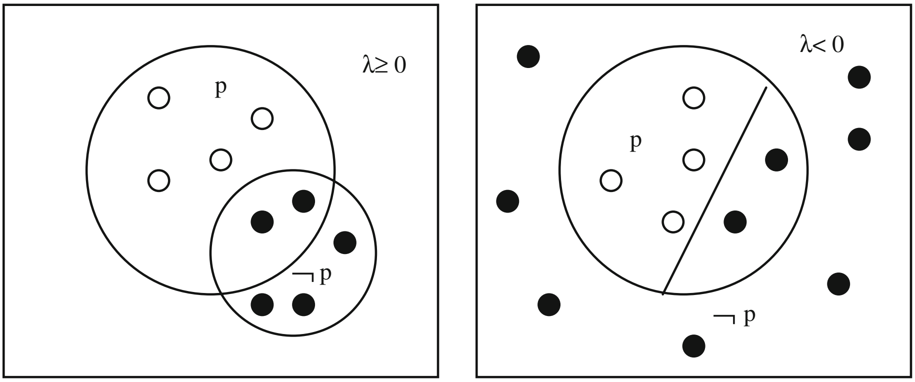


Fig. 6 Example of the (t, f) evolutions by agents



if $\lambda \geq 0$ then $\mu(\neg p) + \mu(p) \leq 1$
if $\lambda < 0$ then $\mu(\neg p) + \mu(p) > 1$

if $\lambda < 0$ then $\mu(\neg p) + \mu(p) > 1$

Fig. 7 Example for the Sugeno negation

In Fig. 7 black points show the agents for which $\neg p$ is true and white point show agents for which p is true. We can see the situations with $\lambda > 0$, and $\lambda < 0$, respectively.

In the evidence theory any subset $A \subseteq U$ of the universe U is associated with a value $m(A)$ called a basic assignment probability such that

$$\sum_{k=1}^{2^N} m(A_k) = 1.$$

Respectively, the belief Bel(Ω) and plausibility Pl(Ω) measures for any set Ω are defined as

$$\mathrm{Bel}(\Omega) = \sum_{A_k \subseteq \Omega} m(A_k), \quad \mathrm{Pl}(\Omega) = \sum_{A_k \cap \Omega \neq \emptyset} m(A_k).$$

Thus, Bel measure includes all subsets that are inside Ω, The Pl measure includes all sets with non-empty intersection with Ω. In the evidence theory as in the probability theory we associate one and only one agent with an element. Therefore, we have Fig. 8.

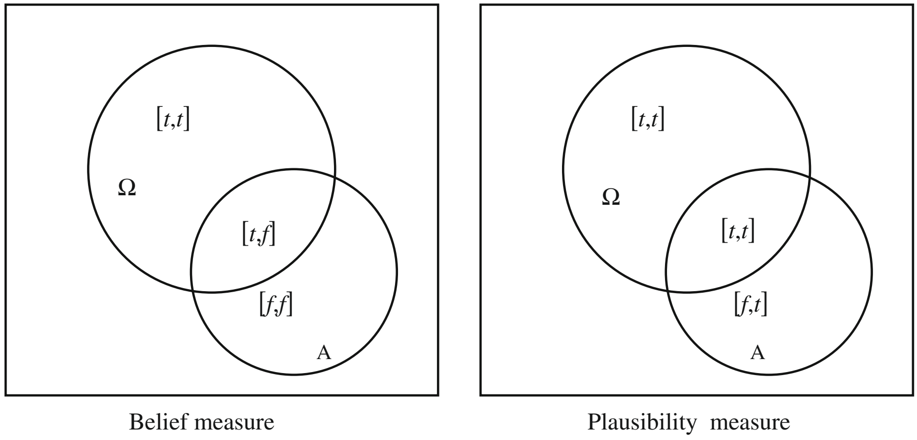


Fig. 8 Example of irrational agents for the belief theory

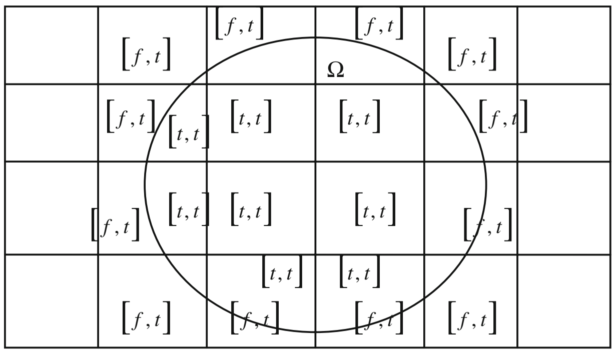


Fig. 9 Example of irrational agents for the rough sets theory

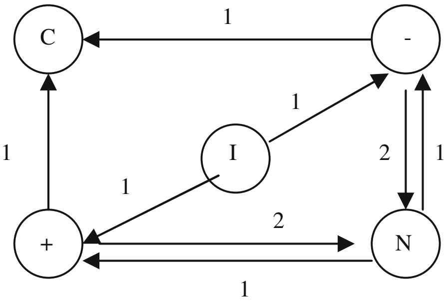
Figure 8 shows set A at the border of the set Ω, which is divided in two parts: one inside Ω and another one outside. For the belief measure we exclude the set A, thus we exclude the false state (f, f) and a logical self-conflicting state (t, f). But for the plausibility measure we accept the (t, t) and the self-conflicting state (f, t). Thus, in the belief measure we exclude any possible self conflicting states, but in the plausibility measure we accept self conflicting states.

There are two different criteria to compute the belief and plausibility measures in the evidence theory. For belief C_1 criterion is related to set Ω, thus C_1 is true for the cases inside Ω. The second criterion C_2 is related to set A, thus C_2 is false for the cases inside A. Now we can put in evidence the logically self-conflicting state (t, f) where we eliminate A also if it is inside Ω. The same is applied to the plausibility, where C_2 is true for cases inside A. In this situation, we accept a logically self-conflicting situation (f, t), where cases in Ω are false but cases in A are true.

We also use the logical self-conflict to study the rough set as we show in Fig. 9.

Figure 9 shows set Ω that includes the logically self-conflicting states (f, t).

Fig. 10 Modeling Montero's voting system [23] by three logically self-conflicting states



With a set of criteria such as $C_1, C_2, \ldots, C_N$ we may have the logically self-conflicting states

$$\begin{bmatrix} \text{state/criteria} & C_1 & C_2 & \ldots & C_N \\ \text{state}_1 & \text{true} & \text{true} & \ldots & \text{true} \\ \text{state}_2 & \text{false} & \text{true} & \ldots & \text{true} \\ \text{state}_3 & \text{true} & \text{false} & \ldots & \text{true} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \text{state}_{2^N} & \text{false} & \text{false} & \ldots & \text{false} \end{bmatrix}.$$

Below we formulate a logic dynamic system with self-conflict defined by the model S, that formalizes in our terms Montero [23] transition diagrams from ignorance to True, False, and Conflict,

$$S = \, <\text{State, Input, Output, State} \times \text{Input} \rightarrow \text{State, State} \rightarrow \text{Output}>,$$

here "State" is a set of logically self-conflicting states, "Input" is a set of the inputs that control the change of the states, Output is a set of the outputs associated with states, State $\times$ Input $\rightarrow$ State is the transition function of the states, and State $\rightarrow$ Output is the output function. We model Montero's voting system [23] as shown in Fig. 10 by three logically self-conflicting states, conflict, ignorance and neutral, C = (true, false, false), I = (false, true, false), Neutral = (false, false, true).

Figure 10 shows a transition graph of the states of the agents in the voting system with a set of inputs $(1, 2)$. In [23] C is defined as the contradiction state, I is defined as the ignorance state, $+$ vote in favor of one party, $-$ vote in favor of the other party and N is the neutral state in which the voter is not in favor of one party or of the other party. When the input is 1, from the ignorance we can move to $+$ or to $-$ and then go to the contradiction. From the neutral situation N we can go to one party or another party. For the input 2 from one party or the other we can come to the neutral state.

5 Conclusion

The hierarchy of sets of agents relative to levels of conflicts, self-conflicts, and irrationality proposed in this paper is intended to provide a base for several areas of further studies. These studies include the foundation of probability theory, fuzzy logic, and other types of uncertainties, as well as real-world interpretation of these theories.

The major weakness in fuzzy logic is its ad hoc nature with operations that are not justified in advance (as it is done in the probability theory), but adjusted in many different ways (by using many different t-norms and t-conorms). The novelty of this paper is not in the introduction of new formulas for the initial membership function values and operations with them, but in the creation of a base for their operational "physical" interpretation by means of conflicting and irrational agents as an internal part of the theory. Physics provides plenty of positive examples of such interpretations that can be inspirational examples for building comprehensive "physical" logics of uncertainty for agents.

A society of agents always has a degree of conceptual conflicts and any agent can be self-conflicting. This affects the agents' abilities to use resources for identifying goals and acting. In this paper, we have established the fundamental logical structure that can be used for making logical decisions in a conflicting multi-agent environment. We expect that in near future this will be an active area of new fundamental research and discoveries.

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